Chapter 4.

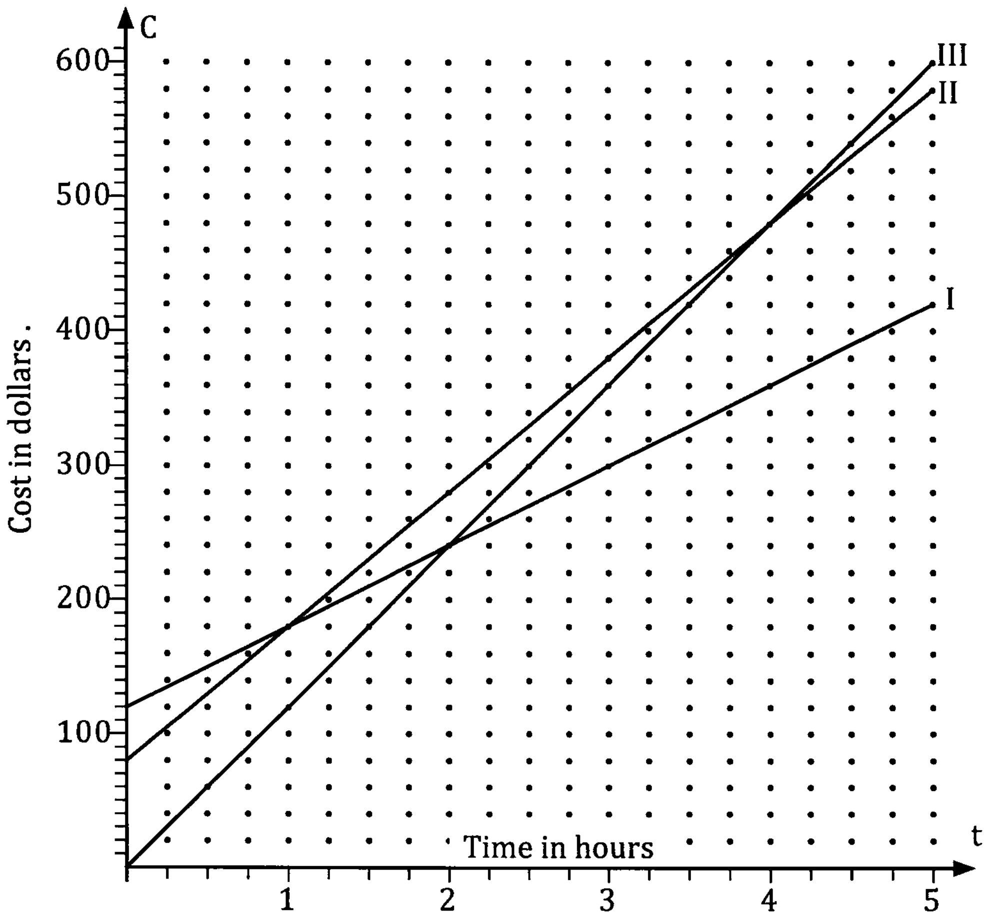
Linear functions.

Situation.

Three electricians, Sparky, Flash and Voltman, have different ways of calculating a customer's bill.

- Sparky charges a standard rate per hour and has no other charges.
- Flash has a fixed, or "standing" charge and then charges a certain amount per hour on top of that.
- Voltman has a higher standing charge than Flash but then charges less per hour.

These three methods are shown graphed below:



- Which line, I, II or III, corresponds to (a) Sparky, (b) Flash, (c) Voltman?
- Ignoring the standing charges who charges most per hour?
  What feature of the graph shows this?
- With the charge being \$C and the time being t hours the equation of line I is:

$$C = 60t + 120.$$

  Determine the equations of lines II and III.
- If you were considering using one of the three electricians for a job and wanted to keep the cost to a minimum which of the three could you dismiss from your considerations?

It is likely that you have already encountered straight line graphs in your mathematical studies of earlier years. Indeed the preliminary work at the beginning of this text included brief mention of the equation

$$y = mx + c.$$

Reading through the next few pages, and then working through exercise 4A which follows, should revise these ideas and provide further practice.

Revision of straight line graphs.
For each of the three electricians in the situation on the previous page the amount they charged the customer, \$C, and the time of the job, t hours, were ***linearly related***. I.e. graphing paired values of the two variables t and C gave a *straight line* in each case.

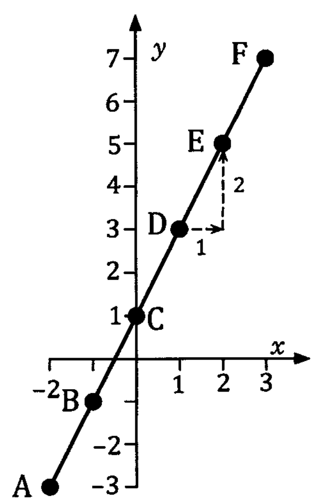
Two important features of straight lines graphs are:

- The steepness or gradient of the line.
  In the situation on the previous page this feature indicated the hourly rate charged by each electrician (neglecting the standing charge).

and

- The point where the line cuts the vertical axis.
  In the situation on the previous page this feature indicated the standing charge for each electrician.

Consider these two features for the graph shown on the right:



- Each time we move to the right 1 unit the line goes up 2 units. We say that the line has a *gradient*, or *slope*, of 2 units. (If a straight line goes *down* for each unit we move right we say it has a *negative* gradient.)

- The line cuts the vertical axis at the point with coordinates (0, 1).

Looking at points A, B, C, D, E and F, all lying on the line, the following table can be created:

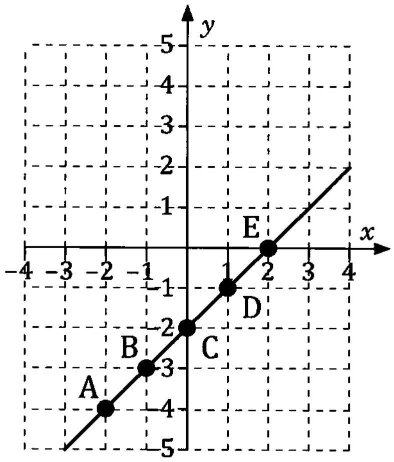
	A	B	C	D	E	F
x coordinate	−2	−1	0	1	2	3
y coordinate	−3	−1	1	3	5	7

Notice that as the x values in the table increase by 1, the y values increase by 2, as we would expect for a table of values for a line with a gradient of 2. If for each unit increase in x the y values did not increase by a constant amount the points would not lie in a straight line.
From the table we see that the rule or equation governing these pairs of numbers is

$$y = 2x + 1$$

> If a straight line cuts the y-axis at (0, c) and has a gradient of m then its equation is : $y = mx + c$

For example:

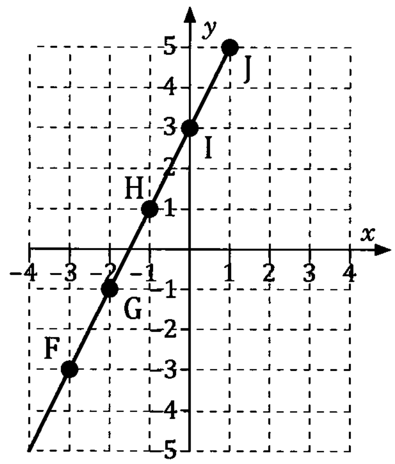


Gradient of line = 1

Cuts y-axis at (0, −2)

Rule: $y = 1x - 2$
Which agrees with
the following table:

	A	B	C	D	E
x	−2	−1	0	1	2
y	−4	−3	−2	−1	0

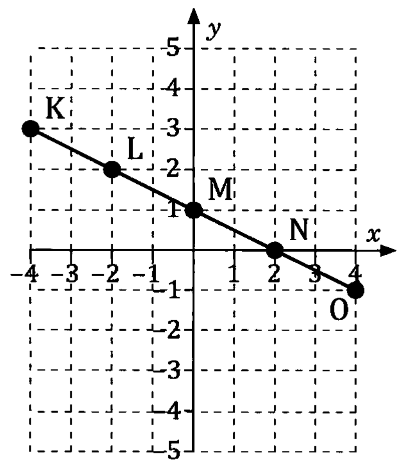


Gradient of line = 2

Cuts y-axis at (0, 3)

Rule: $y = 2x + 3$
Which agrees with
the following table:

	F	G	H	I	J
x	−3	−2	−1	0	1
y	−3	−1	1	3	5



Gradient of line = −0·5

Cuts y-axis at (0, 1)

Rule: $y = -0{\cdot}5x + 1$
Which agrees with
the following table:

	K	L	M	N	O
x	−4	−2	0	2	4
y	3	2	1	0	−1

Example 1
State the equation of the straight line that cuts the y-axis at the point (0, 5) and has a gradient of 7.

A line with gradient m and cutting the y-axis at (0, c) has equation $y = \text{m}x + \text{c}$.
Thus the given line has equation $y = 7x + 5$.

Example 2
A straight line has equation $y = 3x - 5$. Determine its gradient and the coordinates of the point where it cuts the y-axis.

A line with equation $y = \text{m}x + \text{c}$ has gradient m and cuts the y-axis at (0, c).
Thus the given line has gradient 3 and cuts the y-axis at the point (0, −5).

If the equation of a straight line is not presented in the form $y = \text{m}x + \text{c}$ some initial rearrangement may be made to present it in this form, as the next example shows.

Example 3
A straight line has equation $3x + 5y = 20$. Determine its gradient and the coordinates of the point where it cuts the y-axis.

Given:	$3x + 5y = 20$
Subtract $3x$ from each side to isolate $5y$:	$5y = -3x + 20$
Divide each side by 5 to isolate y:	$y = -0{\cdot}6x + 4$

This is now of the form $y = \mathrm{m}x + \mathrm{c}$.
Thus the given line has gradient $-0{\cdot}6$ and cuts the y-axis at the point (0, 4).

The equation of a line is like the "membership ticket" for points lying on the line:

The coordinates of every point lying on a straight line will "fit" the equation of the line and every point not lying on the line will not fit the equation.

Example 4
Determine whether or not the point (−2, −8) lies on the line $y = -3x - 14$.

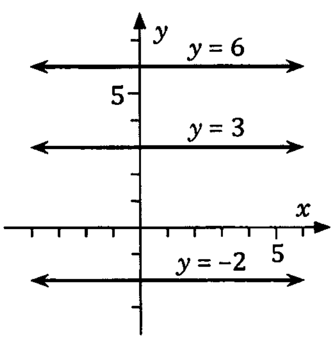
If (−2, −8) lies on the given line then substituting the x-coordinate, −2, into the equation should give the y-coordinate, −8.

If $x = -2$ then
$$\begin{aligned} y &= -3(-2) - 14 \\ &= 6 - 14 \\ &= -8 \end{aligned}$$

Thus the point (−2, −8) <u>does</u> lie on the line $y = -3x - 14$.

Lines parallel to the axes.
Lines parallel to the x-axis have zero gradient.
They will have equations of the form $y = 0x + \mathrm{c}$
i.e. $y = \mathrm{c}$

Hence the graph on the right shows the lines
$y = 6$,
$y = 3$,
and $y = -2$.



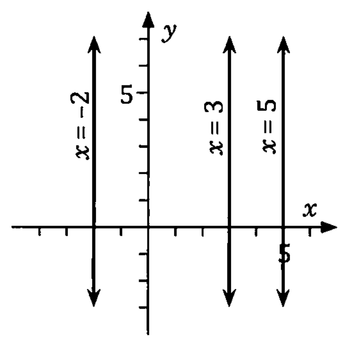
Lines parallel to the y-axis.
Lines parallel to the y-axis have an undefined gradient – we cannot find the vertical rise in the line for each horizontal unit increased because the line rises vertically for zero increase horizontally! Hence we should not expect the rules for vertical lines to be of the form $y = \mathrm{m}x + \mathrm{c}$ because the gradient, m, is undefined. Indeed straight lines parallel to the y-axis are the only straight lines having rules that are <u>not</u> of the form $y = \mathrm{m}x + \mathrm{c}$.

Lines parallel to the y-axis have rules of the form

$$x = c.$$

The graph on the right shows the vertical lines:

$$x = -2,$$
$$x = 3,$$
and $$x = 5.$$



Even though these vertical lines have rules of a different form, points lying on each line must still "obey" the rule. For example, for a point to lie on $x = 3$ the point must have an x-coordinate equal to 3.

Direct proportion and straight line graphs.

As we were reminded in the Preliminary Work section at the beginning of this text, if x and y are **directly proportional** (also called **direct variation**) then

if x is doubled, y is doubled

if x is trebled, y is trebled etc.

If x and y are directly proportional, and if when $x = 1$, $y = \text{k}$, the following table would result:

x	1	2	3	4	5	6
y	k	2k	3k	4k	5k	6k

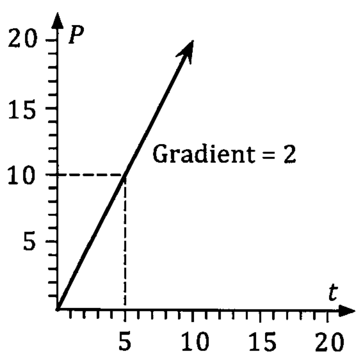
Difference	k	k	k	k	k

As the x values increase by 1 the y values show a constant difference pattern of k.

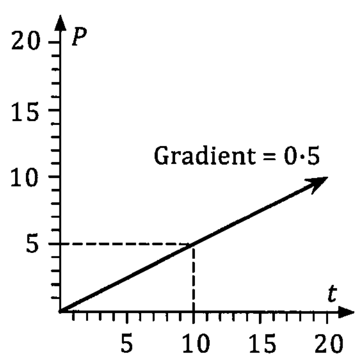
Thus the rule will be of the form $\quad y = \text{k}x + \text{c}.$

From the values in the table $\quad c = 0.$

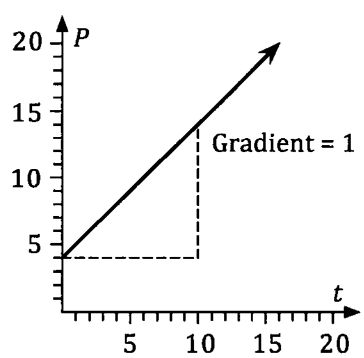
Hence, for direct proportion, the rule relating x and y is $y = \text{k}x$, i.e. a straight line of gradient k and passing through the point (0, 0).



Direct proportion
Rule: $P = 2t$



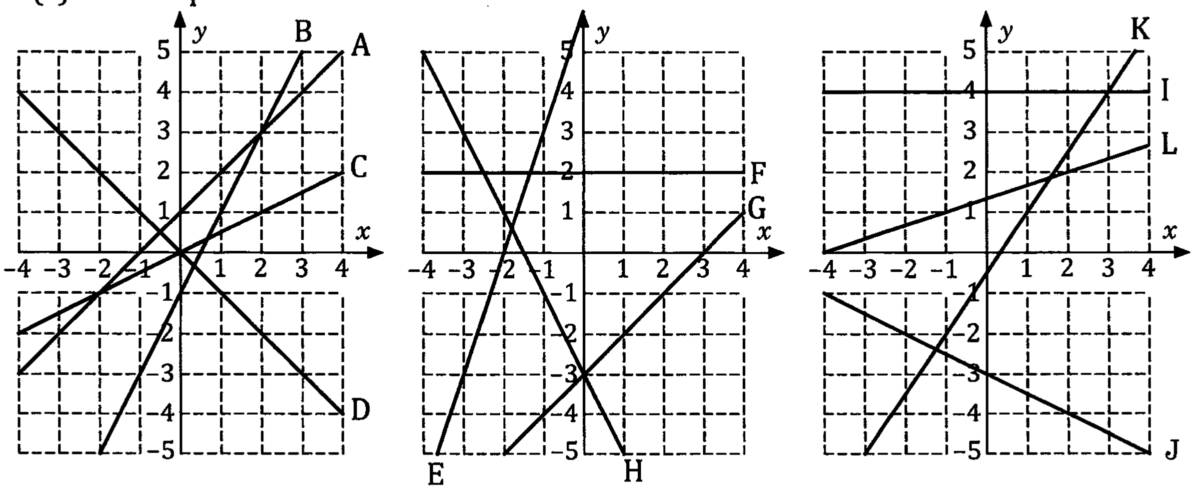
Direct proportion
Rule: $P = 0{\cdot}5t$



Not direct proportion
Rule: $P = t + 4$

Exercise 4A

1. For each of the lines A to L shown below write down
   (a) the coordinates of the point where the line cuts the y-axis.
   (b) the gradient of the line,
   (c) the equation of the line.



2. For each of the following tables determine whether the paired values (x, y) would, if graphed, give points that lie in a straight line. For those that do determine the equation of the straight line. (You should be able to do this question from the tables and should not need to plot the points.)

(a)

x	1	2	3	4	5	6
y	7	9	11	13	15	17

(b)

x	1	2	3	4	5	6
y	-2	3	8	13	18	23

(c)

x	1	2	3	4	5	6
y	0	1	3	6	10	15

(d)

x	1	2	3	4	5	6
y	-3	-2	-1	0	1	2

(e)

x	0	1	2	3	4	5
y	10	8	6	4	2	0

(f)

x	0	1	2	3	4	5
y	5	5	5	5	5	5

(g)

x	3	4	1	6	2	5
y	9	4	16	-9	13	-2

(h)

x	4	2	1	6	3	5
y	7	-3	-8	17	2	12

3. Copy and complete the following table.

Equation	Gradient	y-axis intercept
$y = 2x + 3$	?	(0, ?)
$y = 3x + 4$	?	(0, ?)
$y = -2x - 7$	?	(0, ?)
$y = 6x + 3$	?	(0, ?)

4. Write down the equation of the straight line with gradient 4 and cutting the y-axis at (0, 6).

5. Write down the equation of the straight line cutting the y-axis at (0, –5) and with gradient –1.

6. Suppose that a particular "family" of straight lines are all those with a gradient of 2. Which of the following straight lines are in this family?

Line A:	Equation	$y = 3x + 2$	Line B:	Equation	$y = 2x - 3$
Line C:	Equation	$y = 2$	Line D:	Equation	$y = 2x$
Line E:	Equation	$y = 5 + 2x$	Line F:	Equation	$2y = 4x + 7$
Line G:	Equation	$y - 2x = 5$	Line H:	Equation	$3y + 6x = 5$

7. Suppose that a particular "family" of straight lines are all those passing through the point (0, 6). Which of the following straight lines are in this family?

Line A:	Equation	$y = 5x + 6$	Line B:	Equation	$y = 6x + 5$
Line C:	Equation	$y = 6x$	Line D:	Equation	$y = 6$
Line E:	Equation	$y = 6 + x$	Line F:	Equation	$y + 6 = x$
Line G:	Equation	$2y = -x + 12$	Line H:	Equation	$x + y = 6$

8. Write down the equation of the straight line with gradient –4 and cutting the y-axis at (0, –3). Does this line pass through the point (–1, 1)?

9. Write down the equation of the straight line cutting the y-axis at (0, –3) and with gradient 2. Which of the following points lie on this line ?
A (5, 7), B (–3, –1), C (–0·5 , –4), D (2·5, 2), E (–2, –1).

10. Copy and complete the following table.

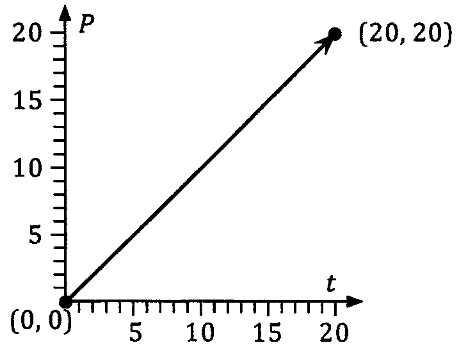
Equation	Written as $y = mx + c$	Gradient	y-axis intercept
$2y = 4x - 5$		?	(0, ?)
$4y = 3x + 7$		?	(0, ?)
$3y - 2x = 6$		?	(0, ?)
$4x + 3y - 6 = 0$		?	(0, ?)
$3x + 5y = 8$		?	(0, ?)

11. Points A (3, a), B (5, b), and C (c, –9) all lie on the line $y = 7x + 5$. Find a, b and c.

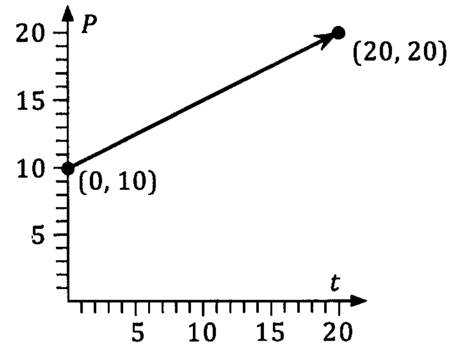
12. The points D, E, F, G, H and I, whose coordinates are given below, all lie on the line $y = dx - 5$. Determine the values of d, e, f, g, h and i.
D (4, –3), E (8, e), F (–2, f), G (13, g), H (h, – 4·5), I (i, –7·5).

13. For each of the following graphs state whether P and t vary directly with each other (i.e. are directly proportional to each other) or not and, for those cases when direct proportion is involved, find the rule for the relationship.

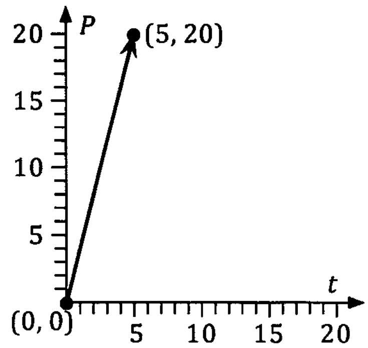
(a)



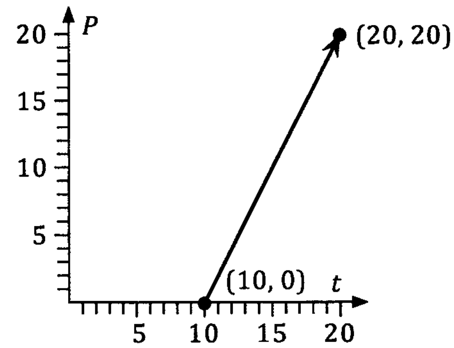
(b)



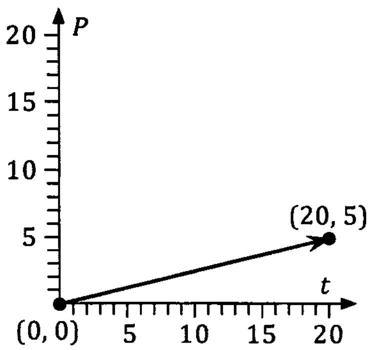
(c)



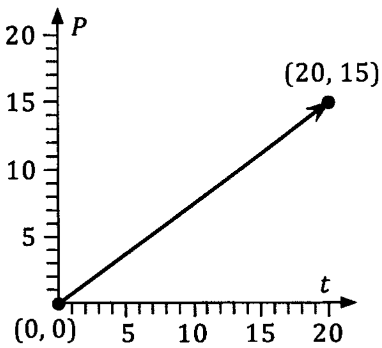
(d)



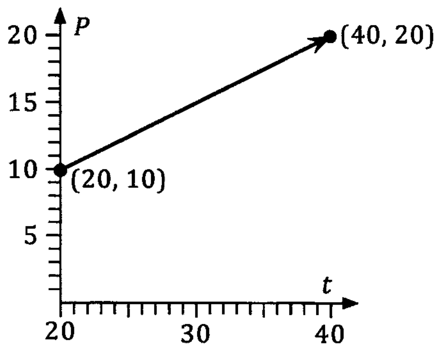
(e)



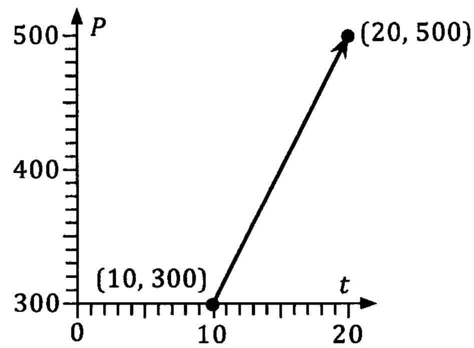
(f)



(g)



(h)



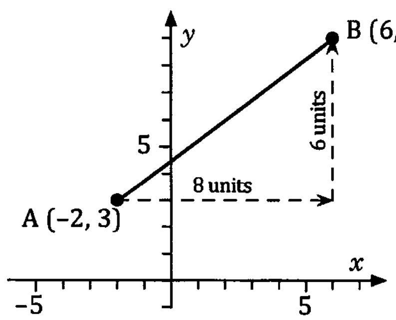
Further considerations.

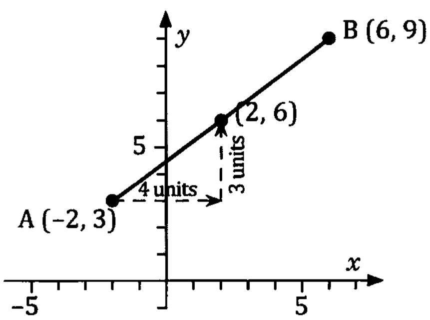
Our understanding of coordinates, gradients and the Pythagorean theorem allows us to:

- Determine the coordinates of the mid-point of a line joining two points.
- Determine the gradient of a line joining two points.
- Determine the distance between two points.

Consider moving from the point A (-2, 3) to B (6, 9). This involves moving right 8 units and up 6 units, as shown below left.

If, from point A, we only wanted to move half way towards point B, we would move just 4 units right and 3 units up. This would take us to the point (2, 6), as shown below right.





Note that as you may have expected, the x-coordinate of the midpoint of the line joining points A and B is the mean of the x-coordinates of the two points, and the y-coordinate of the mid-point is the mean of the y-coordinates of the two points. Hence, if we want to avoid having to plot the points on a graph the following result can be used directly:

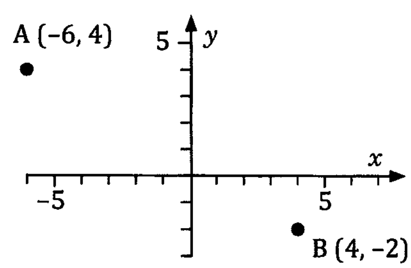
The coordinates of the midpoint of the line joining point A to point B will be:

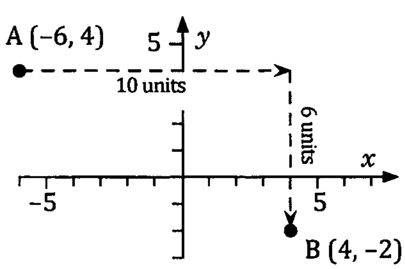
(the mean of the two x coordinates, the mean of the two y-coordinates).

Thus if A has coordinates (x_1, y_1) and B has coordinates (x_2, y_2) then the coordinates of the midpoint will be given by:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Now consider the points A (-6, 4) and B (4, -2) shown graphed below left. To move from A to B we move right 10 units and down 6 units, as shown below right.





Hence in moving right one unit we move down 0·6 units. Thus the gradient of the line through A and B is -0·6.

If we want to avoid having to plot the points on a graph the following result can be used directly:

If a line passes through two points, A and B, then the gradient of the line is:

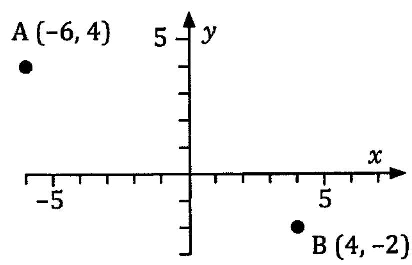
$$\frac{\text{the change in the } y\text{-coordinate in going from A to B}}{\text{the change in the } x\text{-coordinate in going from A to B}}.$$

Thus if A has coordinates (x_1, y_1) and B has coordinates (x_2, y_2) then:

Gradient of the straight line through A (x_1, y_1) and B (x_2, y_2) $= \dfrac{y_2 - y_1}{x_2 - x_1}$.

Note • In the previous formula $\dfrac{y_1 - y_2}{x_1 - x_2}$ would also give the correct answer but $\dfrac{y_1 - y_2}{x_2 - x_1}$ and $\dfrac{y_2 - y_1}{x_1 - x_2}$ would not.

To find the length of the straight line joining A (−6, 4) to B (4, −2) we use the Pythagorean theorem:



$$\begin{aligned} AB^2 &= 10^2 + 6^2 \\ &= 100 + 36 \\ &= 136 \\ AB &= \sqrt{136} \\ &= 11{\cdot}7 \text{ units} \qquad \text{(to 1 dp)} \end{aligned}$$

The length of the line joining A (−6, 4) to B (4, −2) is 11·7 units, to one decimal place.

Again, if we want to apply a formula and not plot the points:

The length of the line joining point A, (x_1, y_1), to point B, (x_2, y_2), is:

$$\sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}\ .$$

I.e. $\sqrt{(\text{change in the } y\text{-coordinates})^2 + (\text{change in the } x\text{-coordinates})^2}$

Example 5

Find the coordinates of the midpoint of AB given that point A has coordinates (−7, 19) and point B has coordinates (5, −3).

Coordinates of midpoint $= \left(\dfrac{-7+5}{2}, \dfrac{19+(-3)}{2}\right)$ i.e. (−1, 8).

The midpoint of the line AB has coordinates (−1, 8)

Example 6

Find the gradient of the straight line through C (3, –5) and D (6, 4).

$$\text{Gradient} = \frac{(4)-(-5)}{(6)-(3)} = \frac{9}{3} = 3$$

The straight line through C and D has a gradient of 3.

Example 7

Find the length of the straight line joining point E (1, –7) to point F (13, –2).

$$EF^2 = (-7 - -2)^2 + (1 - 13)^2$$
$$= (-5)^2 + (-12)^2$$
$$= 169$$
$$EF = \sqrt{169}$$
$$= 13 \text{ units}$$

The length of the line joining E (1, –7) to F (13, –2) is 13 units.

Your calculator, and various internet sites, may have programmed routines that allow

- the coordinates of the midpoint of a line joining two points,
- the gradient of the straight line through two points,

and

- the distance between two points,

to be determined.

Such routines can be useful but make sure that you understand the underlying ideas as shown in the previous examples and can apply them without the assistance of calculator and internet programs if required.

Exercise 4B

1. Calculate the coordinates of the midpoint of the straight line joining each of the following pairs of points.

(a)	(4, 6) and (10, 12)	(b)	(6, 7) and (4, 13)	(c)	(4, 5) and (2, 5)
(d)	(–6, 7) and (2, –5)	(e)	(0, 5) and (–4, 2)	(f)	(5, 3) and (19, –1)
(g)	(6, –2) and (10, –9)	(h)	(–5, 12) and (5, 3)	(i)	(–6, 8) and (8, –6)

2. Find the gradient of the straight line through each of the following pairs of points.

(a)	(4, 6) and (2, 2)	(b)	(6, 7) and (7, 3)	(c)	(4, 5) and (2, 1)
(d)	(6, 7) and (2, 5)	(e)	(5, 3) and (1, 4)	(f)	(3, 3) and (4, 2)
(g)	(4, 3) and (2, 7)	(h)	(5, 2) and (3, –3)	(i)	(4, 2) and (–2, –1)

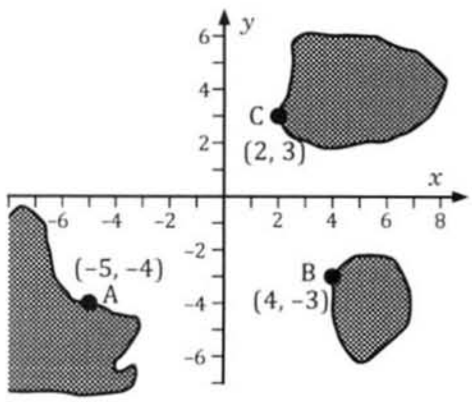
3. Calculate the length of the straight line joining each of the following pairs of points.
   (a) (4, 6) and (7, 10) (b) (6, 7) and (3, 11) (c) (4, 5) and (−8, 10)
   (d) (6, 1) and (−1, 25) (e) (5, −3) and (−3, 12) (f) (−6, 8) and (0, 0)
   (g) (4, 3) and (11, 4) (h) (5, 2) and (−2, 5) (i) (9, 9) and (3, 4)

4. Points C and D have coordinates (3, 6) and (4, 8) respectively. Find
   (a) the gradient of the straight line joining C and D,
   (b) the length of the straight line joining C and D,
   (c) the coordinates of the midpoint of the line CD.

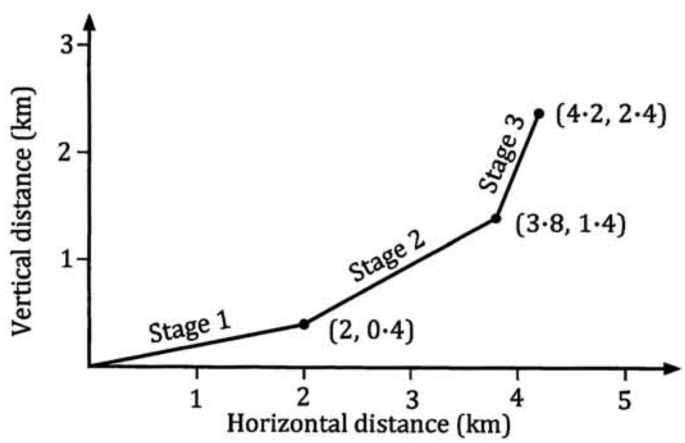
5. Points E and F have coordinates (−1, 1) and (4, 9) respectively. Find
   (a) the gradient of the straight line joining E and F,
   (b) the length of the straight line joining E and F,
   (c) the coordinates of the midpoint of the line EF.

6. The length of the straight line joining point A (1, 4) to point B (7, c) is 10 units. Find the two possible values of c.

7. The diagram on the right shows the location of point A on the mainland and B and C on islands. Each unit shown on the graph is 1 km. Calculate the distance from
   (a) A to B,
   (b) A to C,
   (c) B to C.



8. The diagram below shows a simplified model of the three stages of a proposed mountain climb.



According to this linear model what is the gradient of each stage ?

Determining the equation of a straight line.

As we have already seen, knowing the gradient of a straight line and the coordinates of the point where the line cuts the vertical axis we can determine the equation of the straight line. Similarly we have seen that we can determine the equation of a straight line given the graph of the line or the table of values for the line. Two other common situations that may occur and for which the given information allows us to determine the equation of the line are as follows:

- Given the gradient of the line and the coordinates of *any* point on the line, not necessarily the vertical intercept. (See example 8 below.)
- Given the coordinates of any two points that lie on the line. (See example 9 below.)

Example 8 (Given the gradient and one other point on the line.)

Find the equation of the straight line through the point (4, −3) and with a gradient of −2.

A straight line of gradient m has an equation of the form $y = mx + c$.

Thus the given line will have an equation of the form $y = -2x + c$.

The line passes through the point (4, −3).

Thus the values $x = 4$ and $y = -3$ must "fit" the equation, i.e. $(-3) = -2(4) + c$

giving $c = 5$.

Thus the given line has equation $y = -2x + 5$.

Example 9 (Given two points that lie on the line.)

Find the equation of the straight line through the points (−2, 8) and (4, −1).

First we determine the gradient of the line, either by reasoning:

Starting with the point with the lower x-coordinate (−2, 8), and moving to the other point, (4, −1), we travel across 6 units and down 9 units. Thus in moving across 1 unit we travel <u>down</u> $\frac{9}{6}$ units, i.e. $\frac{3}{2}$ units. The gradient of the line is $-\frac{3}{2}$.

Or by use of $\frac{y_2 - y_1}{x_2 - x_1}$: Gradient $= \frac{8-(-1)}{-2-4}$

$= -1{\cdot}5$

Thus the given line will have an equation of the form $y = -1{\cdot}5x + c$.

The line passes through the point (4, −1). Thus $-1 = -1{\cdot}5(4) + c$

giving $c = 5$.

Thus the given line has equation $y = -1{\cdot}5x + 5$.

(The reader should confirm that using the point (−2, 8) and saying that the values $x = -2$ and $y = 8$ must "fit" the equation also gives $c = 5$.)

Example 10 **(Given information that allows us to determine two points that lie on the line and then proceed as in the previous example.)**

The line $y = 2x - 6$ cuts the x-axis at the point A.
The line $y = 5 - 7x$ passes through the point B, coordinates (−1, k).
Find the equation of the straight line through points A and B.

Any point on the x-axis has a y-coordinate of zero. Thus at point A: $0 = 2x - 6$

$\therefore \quad x = 3$

Point A has coordinates (3, 0).

Point B, (−1, k), lies on the line $y = 5 - 7x$. Thus $k = 5 - 7(-1)$

$\therefore \quad k = 12$

Point B has coordinates (−1, 12).

The line through points A (3, 0) and B (−1, 12) has gradient $\dfrac{12-0}{-1-3} = -3$

Thus the required equation is of the form $y = -3x + c$

But the point (3, 0) lies on this line, hence: $0 = -3(3) + c$

$\therefore \quad c = 9$

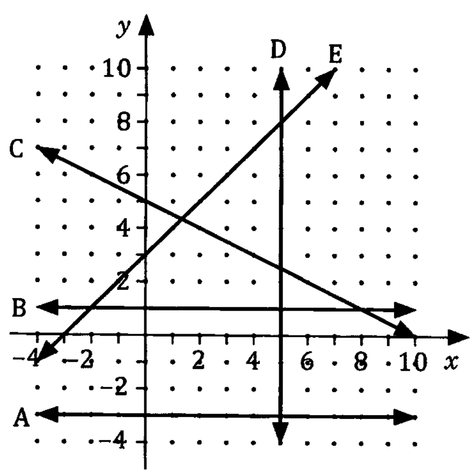
The required equation is $y = -3x + 9$.

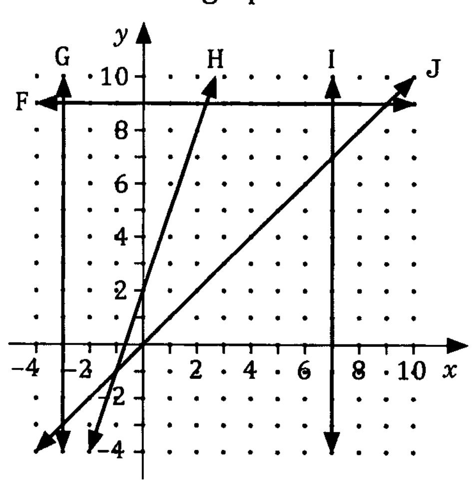
Calculator and internet routines.

As with determining the coordinates of the midpoint of a line joining two points,
the distance between two points,
and the gradient of the straight line through two points,
your calculator, and some internet sites, may also have programmed routines that allow the equation of a line to be determined simple by inputting the coordinates of two points on the line, or inputting the gradient and the coordinates of just one point on the line. Again such routines can be useful but make sure that you understand the underlying ideas and can apply them without the assistance of such programs if required.

Exercise 4C

1. Write the equations of each of the lines A to J shown in the graphs below.





2. What is the equation of the x axis?

3. What is the equation of the y axis?

4. Write down the equation of the straight line with gradient 3 and cutting the y-axis at (0, 4).
Does this line pass through the point (–1, 1) ?

5. Write down the equation of the straight line cutting the y-axis at (0, 2) and with gradient 0·5.
Which of the following points lie on this line ?
A (2, 1), B (2, 0), C (4, 2), D (–6, –1), E (4, 4).

6. Find the equation of each of the following straight lines.

(a) Gradient 1, through (3, 5).	(b) Gradient –1, through (6, –1).
(c) Gradient –2, through (3, 2).	(d) Gradient 5, through (–2, –2).
(e) Gradient 0·5, through (8, 9).	(f) Gradient – 0·5, through (–3, 0).
(g) Gradient 1·5, through (9, 2).	(h) Gradient $-\frac{1}{3}$, through (7, –1).

7. Find the equation of each of the following straight lines.

(a) Through (2, 5) and (6, 9).	(b) Through (0, –1) and (2, –9).
(c) Through (14, 1) and (16, –5).	(d) Through (1, 1) and (2, 3).
(e) Through (1, 2) and (13, 6).	(f) Through (3, –2) and (–1, 6).
(g) Through (3, 9) and (0, 4).	(h) Through (0, 5) and (2, –5).

8. Find the equation of the straight line passing through (1, 1) and (4, 7).
Determine which of the points listed below lie on this line.
A(7, 15), B(7, 13), C(2, 2), D(–1, 3), E(6, 11).

9. Find the equation of the straight line with gradient 0·5 and passing through the point (3, 4).
Given that each of the points listed below lie on this line determine the values of f, g, h, i and j.
F(9, f), G(–9, g), H(h, 9), I(i, 1·5), J(3·8, j).

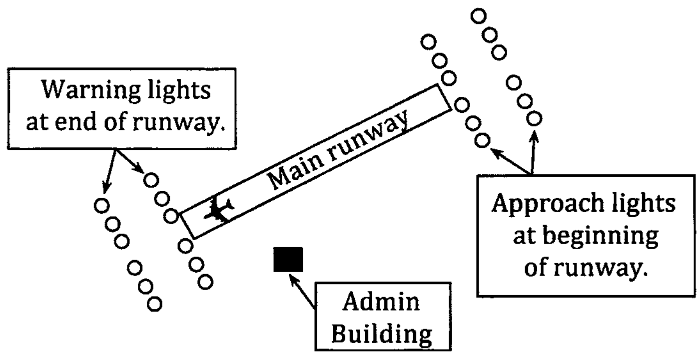
10. Find the coordinates of the point where the line $2y = x - 4$ cuts the x-axis.
Find the equation of the straight line through this point and (–1, 10).

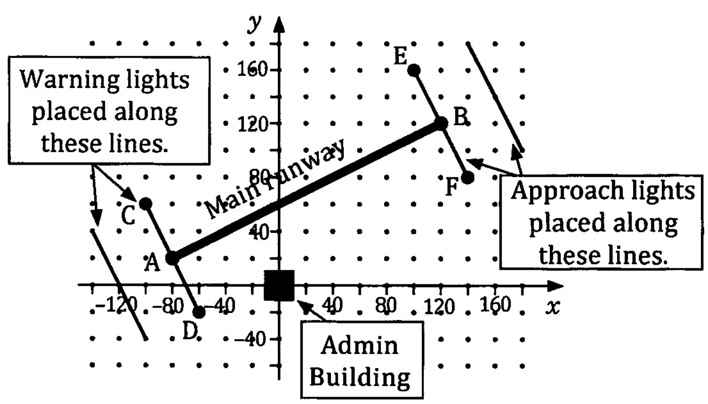
11. Find the coordinates of the point where the line $2y = -x + 6$ cuts the x-axis.
Find the equation of the straight line through this point and the point (8, 8).

12. If we plot degrees Centigrade, (°C), on the x-axis and degrees Fahrenheit, (°F), on the y-axis, the graph for converting from one scale to the other is a straight line. Given that 100°C is the same as 212°F and 50°C is the same as 122°F find the equation of the line in the form $F = mC + b$, where m and b are constants.
Convert the following to °F. (a) 55°C (b) 125°C (c) −10°C
Convert the following to °C. (d) 59°F (e) 86°F (f) −40°F

13. If we plot the "Number of metered units", N, on the x-axis and the "Amount to be paid", $\$A$, on the y-axis then the graph for calculating a telephone bill is a straight line.
If the bill for 100 units is \$64 and for 175 units is \$82, determine the equation of this line in the form $A = mN + c$, where m and c are constants.

14. The diagram on the right shows the proposed layout of a small airfield. The diagram shows the main runway, the approach lights, the warning lights and the administration building.
The second diagram shows the proposal as a graph with lengths in metres and the admin building as the origin.
Find:





(a) the coordinates of the points A, B, C, D, E and F, (all divisible by 20).
(b) the length AB,
(c) the equation of the straight line through A and B,
(d) the equation of the straight line through C and D,
(e) the equation of the straight line through E and F.

15. At 8 a.m. one morning an industrial fuel tank contains 4000 litres of fuel.
Fuel is being withdrawn from the tank at a constant rate of 0·25 litres per minute.
Writing A litres for the amount of fuel in the tank at time t hours past 8 a.m. find A when $t = 2$ and t when $A = 3850$.
Express the relationship between A and t in the form $A = mt + c$.

16. Susie Fuse, an electrician, charges her customers a standard call out fee plus a certain amount per hour. For a job that takes her 3 hours she charges $445 and for a job that takes her 4·5 hours she charges $625.
Write her method of charging in the form $C = \mathrm{m}T + \mathrm{c}$, where $\$C$ is the cost to the customer for a job that takes T hours and m and c are constants.

17. A linear relationship exists between the profit, $\$P$, that the organisers of a concert make, and N, the number of tickets they sell. With P plotted on the vertical, y, axis and N on the horizontal, x, axis the line of this relationship passes through the points (900, 400) and (1100, 1300). Find the equation of this line in the form

$$P = \mathrm{m}N + \mathrm{c},$$

where m and c are constants.

(a) What will be the profit when 1500 tickets are sold?

(b) If the concert hall has a maximum capacity of 2500 what profit will the organisers make if they give away 150 complimentary tickets and sell all the rest?

(c) What is the least number of tickets the organisers could sell and still not make a loss?

18. The owner of a computer shop calculates that his weekly profit from computer sales is linearly related to the number of computers sold that week.
If he sells 10 computers in a week his total profit is $560.
If he only sells 5 computers in the week he makes a profit of $10.
The rule relating his total profit for the week, $\$P$, to the number of computers sold, x, is given by:

$$\text{Total profit in dollars} = \mathrm{m}x - \mathrm{c},$$

$\$c$ being the fixed weekly cost of running the shop.

(a) Calculate m and c.

(b) What is his weekly profit from computer sales in a week that he sells 20 computers?

19. When a particular spring, of unstretched length L_0 metres, has a mass of M kg suspended from one end its new length, L metres, is given by:

$$L = \mathrm{k}M + L_0 \quad \text{where k is a constant.}$$

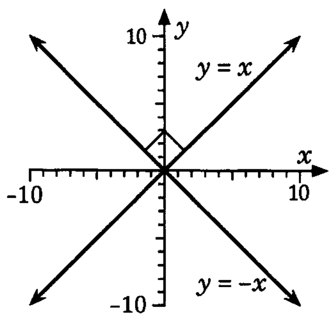
A graph of M plotted on the horizontal axis and L on the vertical axis passes through the points (2, 0·85) and (3, 1·05).

Calculate k and L_0 and hence determine how much the spring is extended **beyond its natural length** when a mass of 250 g is suspended from it.

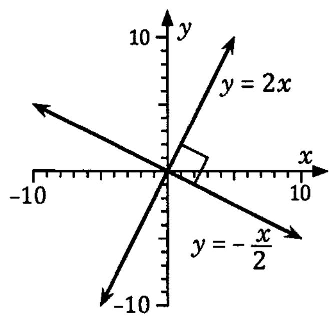
Parallel and perpendicular lines.

Two lines that are parallel must have the same gradient but what is the relationship between the gradients of two perpendicular lines?

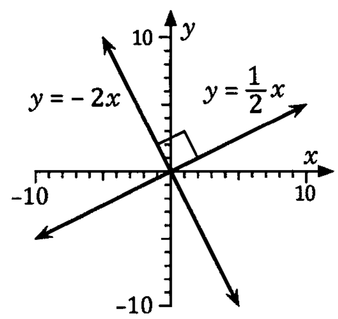
Each of the following diagrams show two perpendicular lines. Can you notice any pattern in the gradients of two lines that are perpendicular?



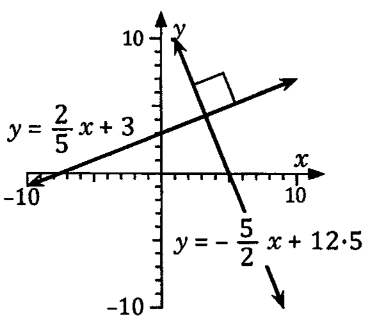
Gradients of 1 and −1.



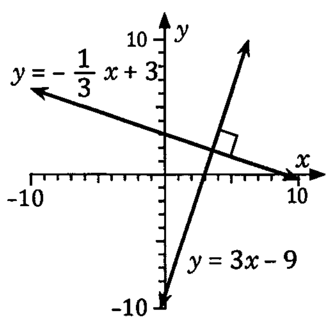
Gradients of 2 and $-\frac{1}{2}$.



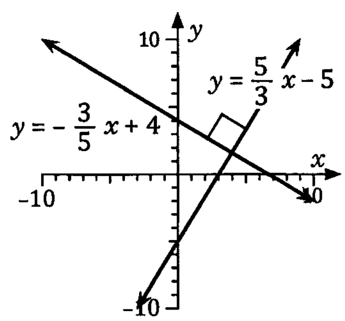
Gradients of −2 and $\frac{1}{2}$.



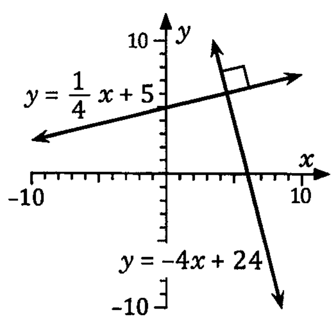
Gradients of $\frac{2}{5}$ and $-\frac{5}{2}$.



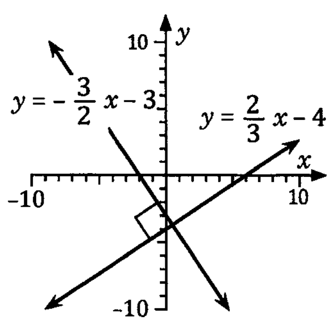
Gradients of $-\frac{1}{3}$ and 3.



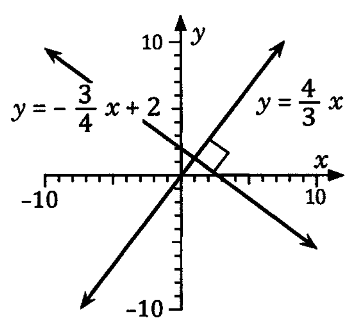
Gradients of $-\frac{3}{5}$ and $\frac{5}{3}$.



Gradients of $\frac{1}{4}$ and −4.



Gradients of $-\frac{3}{2}$ and $\frac{2}{3}$.



Gradients of $-\frac{3}{4}$ and $\frac{4}{3}$.

Did you notice that:

If the gradient of one line is m, a perpendicular line has gradient $-\frac{1}{m}$?

This can be summarised as follows:

The gradients of two perpendicular lines have a product of − 1.

Note • The only situation involving perpendicular lines where this rule does not apply is that of a horizontal line, e.g. $y = 3$ (zero gradient), and a vertical line, e.g. $x = 4$ (infinite gradient). In such cases the product of the gradients does not equal −1 but the lines are perpendicular.

Exercise 4D

1. The eleven lines whose equations are given below contain five pairs of parallel lines and one that is not parallel to any of the others.
   List the five pairs. (i.e. A and E, ? and ?, ...).

A:	$y = 2x + 3$	B:	$y = 3x + 4$	C:	$y = 5x + 3$
D:	$y = \frac{1}{2}x + 3$	E:	$y = 2x - 1$	F:	$y = 5 - \frac{1}{2}x$
G:	$y + 5x = 1$	H:	$y - 5x = 4$	I:	$y = 1 - 5x$
J:	$y = 3x - 2$	K:	$2y + x = 6$		

2. Find the equation of the straight line through the point (−1, −7) and parallel to the line $y = 2x + 3$.

3. The eleven lines whose equations are given below contain five pairs of perpendicular lines and one that is not perpendicular to any of the others.
   List the five pairs.

A:	$y = -2x + 3$	B:	$y = 3x$	C:	$y = x + 3$
D:	$y = \frac{1}{2}x + 1$	E:	$y = -x + 1$	F:	$y = 3$
G:	$3y + x = 3$	H:	$2y = 3x + 2$	I:	$2y + 3x = 8$
J:	$3y = 2x - 9$	K:	$x = -2$		

4. Find the equation of the straight line that is perpendicular to $y = 2x + 3$ and passes through the point (−4, 7).

5. Find the equation of the straight line that is perpendicular to $3y = 5 - x$ and passes through the point (−1, 2).

6. (a) The lines $y = x - 3$ and $y = 3x - 7$ intersect at the point B.
   Using your calculator, or otherwise, find the coordinates of point B.
   (b) Find the equation of the line that is perpendicular to $2y + x = 8$ and that passes through the point B.

Miscellaneous Exercise Four.

This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the preliminary work section at the beginning of the book.

1. Which of the following are equations of straight lines?

   A: $y = 2x + 3$ B: $y = x^2 + 3$ C: $y = 3x - 1$ D: $y = 3 - x^3$

   E: $y = 4 - 3x$ F: $2y = 4x + 5$ G: $y = 2x + 3x^2$ H: $y = 4x$

   I: $x = 4y$ J: $2y + 3x = 5$ K: $y = \frac{2}{x+1}$ L: $y = \frac{x+1}{2}$

2. State whether each of the points A to E lie on the line $y = 3x - 5$.

   A (6, 12) B(5, 11) C (2, 1) D (−3, −13) E (−1, −8)

3. State whether each of the points F to J lie on the line $y = -x + 6$.

   F (1, 5) G(0, 6) H (2, 8) I (−1, 4) J (6, 0)

4. For $f(x) = 2x + 3$ and $g(x) = 5x - 18$ determine each of the following.

   (a) f(4) (b) f(−2) (c) f(10)
   (d) g(2) (e) g(−2) (f) g(6·5)
   (g) f(1) + f(2) (h) g(1) + g(2) (i) f(m) + g(m)
   (j) The value of m given that f(m) = 15.
   (k) The value of p given that g(p) = 7.
   (l) The value of q given that f(q) = g(q).
   (m) The value of r given that f(r) = r.
   (n) The value of s given that g(s) = s.

5. Find where each of the following pairs of lines intersect

   (a) $y = 2x - 11$ and $y = -3x + 4$ (b) $5x + 2y = 3$ and $2x + 3y = 10$

6. State the natural domain and the corresponding range for each of the following.

   (a) $f(x) = x - 5$ (b) $f(x) = \sqrt{x-5}$ (c) $f(x) = (x-5)^2$

   (d) $f(x) = \frac{1}{x-5}$ (e) $f(x) = \frac{1}{(x-5)^2}$ (f) $f(x) = \frac{1}{\sqrt{x-5}}$

7. Prove that points A (29, 16) B (25, 24) C (−2, 33) D (−10, 29), E (−15, −6) F (29, 2)

   (a) all lie on the same circle centre O, coordinates (5, 9),
   (b) are such that OA is perpendicular to OC,
   (c) are such that BE is a diagonal of the circle.

8. Given the relationship between x and y is linear determine the values of $a, b, c, \ldots g$.

x	0	1	2	3	4	5	6	...	f	g
y	a	b	c	14	d	24	e	...	54	494

9. Each vertex of an equilateral triangle of side 10 cm is the centre of a circle of radius 5 cm. Find the area of the central region bounded by the circles giving your answer as an exact value.